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Technical Note TN 22

A note on the summation of some series
of Bessel functions

by

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1. Rutgers [1] has derived explicit expressions for the sum of series of the kind

$$(1.1) \quad S_k(r) = \sum_{n=0}^{\infty} (\nu+2n)^k I_{\nu+2n}(r),$$

where k is a non-negative integer. In a recent paper Carlitz [2] considered these series from a different point of view. Here it will be shown that such series may be summed in a much more simple and elementary way. From Bessel's differential equation it follows at once that

$$(1.2) \quad \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\} I_{\mu}(r) = \mu^2 I_{\mu}(r).$$

Hence the following recurrence relation exists

$$(1.3) \quad S_{k+2}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\} S_k(r).$$

Therefore we need only consider the cases $k=0$ and $k=1$.

In the special case $\nu=0$ we may start from the well-known Fourier expansion

$$(1.4) \quad e^r \cos \varphi = \sum_{n=0}^{\infty} \varepsilon_n \cos n\varphi I_n(r).$$

Expanding both in rising powers of φ it follows at once that the sum of the series

$$(1.5) \quad \sum_{n=0}^{\infty} \varepsilon_n n^{2k} I_n(r)$$

is determined by the coefficient of φ^{2k} in the expansion of $\exp(r \cos \varphi) = \exp\left\{r\left(1 - \frac{1}{2}\varphi^2 + \dots\right)\right\}$ which is obviously of the form $\exp(r \cdot \varphi_k(r))$ where $\varphi_k(r)$ is a polynomial of degree k . A recurrence relation between successive $\varphi_k(r)$ could be obtained by noting that both sides of (1.4) satisfy the Helmholtz equation $(\Delta - 1)f = 0$ where Δ is the Laplacian in polar coordinates (r, φ) . However, it is simpler to use (1.4) for $\varphi=0$ which gives the well-known result

$$(1.6) \quad \sum_{n=0}^{\infty} \varepsilon_n I_n(r) = e^r,$$

and then to apply the recurrence relation (1.3).

This gives

$$(1.7) \quad \sum_{n=0}^{\infty} \varepsilon_n n^{2k} I_n(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\}^k \cdot e^r.$$

Since

$$(1.8) \quad \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\} \cdot e^r \varphi(r) = \\ = e^r \left\{ r^2 \frac{\partial^2}{\partial r^2} + (2r^2 + r) \frac{\partial}{\partial r} + r \right\} \cdot \varphi(r),$$

we may also write^{*)}

$$(1.9) \quad \sum_{n=0}^{\infty} \varepsilon_n n^{2k} I_n(r) = e^r \left\{ r^2 \frac{\partial^2}{\partial r^2} + (2r^2 + r) \frac{\partial}{\partial r} + r \right\}^k \cdot 1.$$

By making the substitution $\varphi = \frac{1}{2}\pi$ in (1.4) we obtain

$$(1.10) \quad \sum_{n=0}^{\infty} (-1)^n \varepsilon_{2n} I_{2n}(r) = 1,$$

and next^{**)}

$$(1.11) \quad \sum_{n=0}^{\infty} (-1)^n \varepsilon_{2n} (2n)^{2k} I_{2n}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\}^k \cdot 1.$$

If (1.4) is differentiated with respect to φ we obtain when substituting $\varphi = \frac{1}{2}\pi$

$$(1.12) \quad \sum_{n=0}^{\infty} (-1)^n \varepsilon_{2n+1} (2n+1) I_{2n+1}(r) = r,$$

and next^{***)}

$$(1.13) \quad \sum_{n=0}^{\infty} (-1)^n \varepsilon_{2n+1} (2n+1)^{2k+1} I_{2n+1}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\}^k \cdot r.$$

If r is replaced by ir we obtain expansions containing Bessel functions of the first kind. E.g. from (1.11) may derive

$$(1.14) \quad \sum_{n=0}^{\infty} \varepsilon_{2n} (2n)^{2k} J_{2n}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 + r^2 \right\}^k \cdot 1.$$

*) Cf. Carlitz l.c. formula (3.4).

**) Ib. formula (6.2).

***) Ib. formula (6.1).

2. More generally we now consider the series

$$(2.1) \quad \sum_{n=0}^{\infty} c_n (\nu+n)^k I_{\nu+n}(r)$$

where again k is a non-negative integer and where the coefficients c_n are given by the power series expansion

$$(2.2) \quad f(t) = \sum_{n=0}^{\infty} c_n t^n .$$

It is sufficient to consider only the case $k=0$.

Since

$$(2.3) \quad \mu I_{\mu}(r) = r I_{\mu-1}(r) - r I'_{\mu}(r)$$

the case $k=1$ may be reduced to the previous one. For larger values of k we may use the recurrence relation (1.3). We shall write

$$(2.4) \quad S(r, \nu, f(t)) = \sum_{n=0}^{\infty} c_n I_{\nu+n}(r).$$

Using Sommerfeld's integral expression

$$(2.5) \quad I_{\mu}(r) = \frac{1}{2\pi i} \int_{-\infty-\pi i}^{-\infty+\pi i} e^{r \operatorname{ch} w + \mu w} dw$$

we obtain without difficulty

$$(2.6) \quad S(r, \nu, f) = \frac{1}{2\pi i} \int_{-\infty-\pi i}^{-\infty+\pi i} e^{r \operatorname{ch} w + \nu w} f(e^w) dw$$

The right-hand side of (2.6) is obviously reducible to a Bessel function for the following particular choice

$$(2.7) \quad e^{\nu w} f(e^w) = \frac{d}{dw} \frac{e^{\nu w}}{1-e^{-2w}},$$

i.e. when

$$(2.8) \quad f(t) = t^{1-\nu} \left(\frac{t^{\nu}}{1-t^2} \right)' = \sum_{n=0}^{\infty} (\nu+2n) t^{2n} .$$

After partial integration it follows that

$$(2.9) \quad S(r, \nu, f) = \frac{1}{2} r I_{\nu-1}(r) .$$

Applying (1.3) we obtain^{*})

$$(2.10) \quad 2 \sum_{n=0}^{\infty} (\nu+2n)^{2k+1} I_{\nu+2n}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\}^k \cdot r I_{\nu}(r).$$

The same technique enables us to find simple expressions for series such as

$$(2.11) \quad \sum_{n=0}^{\infty} \varepsilon_n I_{\nu+n}(r) \quad .$$

In this case we have $f(t) = (1+t)/(1-t)$ so that

$$(2.12) \quad S(r, \nu, \frac{1+t}{1-t}) = \frac{-1}{2\pi i} \int_{-\infty - \pi i}^{-\infty + \pi i} e^{r \operatorname{ch} w + \nu w} \operatorname{cth} \frac{1}{2} w \, dw.$$

A simple calculation shows that

$$(2.13) \quad \left(\frac{\partial}{\partial r} - 1 \right) S = \frac{\nu}{r} I_{\nu}(r)$$

so that

$$(2.14) \quad S = \nu e^r \int_0^r e^{-e} e^{-1} I_{\nu}(e) \, de \quad .$$

The derivation of similar results of this kind may be left to the reader.

References

1. Rutgers, J.G. Extension d'une serie des fonctions de Bessel, I and II. Kon.Ned.Akad.v.Wetensch. Proc. 45, 929-936 and 987-993 (1942).
2. Carlitz, L. Summation of some series of Bessel functions. Ibid. A 65, 47-54 (1962).

^{*}) Ib. formula (5.3).